Some applications of duality for Lévy processes in a half-line

Jean Bertoin, Mladen Savov

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Abstract

The central result of this paper is an analytic duality relation for real-valued Lévy processes killed upon exiting a half-line. By Nagasawa's theorem, this yields a remarkable time-reversal identity involving the Lévy process conditioned to stay positive. As examples of applications, we construct a version of the Lévy process indexed by the entire real line and started from $-\infty$ which enjoys a natural spatial-stationarity property, and point out that the latter leads to a natural Lamperti-type representation for self-similar Markov processes in $(0,\infty)$ started from the entrance point $0+$.

1 Introduction

A celebrated result due to David Williams (cf. Theorem 3.4 in [20]) can be stated as follows. Consider a real Brownian motion $(B^x_t)_{t\geq 0}$ started from some level $x>0$ and $T=\inf\{t\geq 0 : B^x_t \leq 0\}$ its first exit-time from $(0,\infty)$. Then the process $(B^x_{T-t})_{0\leq t<T}$ obtained by time-reversing the Brownian path at time T, has the same distribution as a three-dimensional Bessel process started from 0 and killed at the time of its last-passage at level x. This relation should be viewed as the probabilistic counterpart of an analytic duality between the transition probabilities $q_t(x,\mathrm{d}y)$ of the Brownian motion killed upon exiting $(0,\infty)$ and $p^{\uparrow}_t(x,\mathrm{d}y)$ of the three-dimensional Bessel process. Specifically, there is the identity

$$q_t(x,\mathrm{d}y)x\mathrm{d}x = p^{\uparrow}_t(y,\mathrm{d}x)y\mathrm{d}y\,, \qquad x,y\in(0,\infty). \tag{1}$$

One further observes that the duality measure $x\mathrm{d}x$ on $(0,\infty)$ coincides with the potential measure $U^{\uparrow}(0,\mathrm{d}x)=\int_0^\infty \mathrm{d}t p^{\uparrow}_t(0,\mathrm{d}x)$ of the three-dimensional Bessel process started from 0,

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and the time-reversal identity of Williams then follows from a general result of Nagasawa [17] for Markov processes in duality. We also refer to Azéma [1] and Chung and Walsh [9] for further seminal contributions in this area.

It has been observed in Theorem VII.18 in [2] that similar arguments can be applied to Lévy processes with no positive jumps, and yield an extension of Williams' time-reversal identity in that setting. More precisely, consider a Lévy process ξ with no positive jumps; the role of the Brownian motion is now played by $\hat{\xi} = -\xi$, and that of the three-dimensional Bessel process by $\xi^{\uparrow}$, which should be thought of as ξ conditioned to stay positive (in general such a conditioning is singular and has to be understood in terms of Doob's h-transform). Of course, the absence of positive jumps of ξ is crucial as it ensures that the downwards passages for $\hat{\xi}$ occur continuously. The central result of the present work is that a duality identity extending (1) holds for general Lévy processes (possibly with positive jumps), and as a consequence so does the remarkable time-reversal identity for Lévy processes which do not tend to $-\infty$. A fundamental feature of this extension is the possibility of downwards crossings by a jump for $\hat{\xi}$, so in general the Lévy process $\hat{\xi}$ and the version of ξ conditioned to stay positive, $\xi^{\uparrow}$, have to start from appropriate random locations in $[0, \infty)$.

This duality relation has a number of applications, some of which have already been observed in the literature. In particular, it enables us to construct a process $(\xi_t)_{t\in\mathbb{R}}$ indexed by the real line that fulfills a spatial invariance property and which may be thought of as a version of the Lévy process ξ started from $-\infty$. More precisely, it appears as the limit in distribution as $x \to -\infty$ of the Lévy process started at time 0 from x and shifted in time at the instant of its first entrance in $(0, \infty)$. In this vein, we point at a remarkable representation of positive self-similar Markov processes $(X_t)_{t\geq 0}$ in $(0, \infty)$ started from the boundary point $0+$ as a time-change of $\exp(\xi)$, which extends the classical construction due to Lamperti [16] when X starts from a strictly positive position.

This paper is organized as follows. In the next section we first recall some useful elements of fluctuation theory for Lévy processes, and then present the key duality relation. After discussing the classical weak convergence of the under and over shoots in the framework of renewal theory applied to the ladder height process, we finally use Nagasawa's theorem to establish an identity involving time-reversed processes which provides the probabilistic counterpart of the duality relation. Section 3 is devoted to applications to limit theorems. We first observe that there is a natural version of the Lévy process indexed by the entire real line which enjoys a remarkable spatial-stationarity property. Then we show that this process appears as the limit as $z \to -\infty$ of the genuine Lévy process started from z and shifted in time at its first entrance in $(0, \infty)$. This weak limit theorem encompasses the classical convergence of the under and over shoots, and the combination with Lamperti's transformation points at a simple approach for studying the entrance boundary of positive self-similar Markov processes.

2 Duality and time-reversal in a half-line

2.1 Some notation and preliminaries

We introduce some background on Lévy processes and fluctuation theory that will be needed here, referring the reader to [2], [10] or [14] for a complete account. **We implicitly exclude the compound Poisson processes** (merely to avoid discussing periodicity).

We shall use the canonical notation : the probability space is chosen to be $\Omega = \mathbb{D}([0,\infty),\mathbb{R})$, the space of càdlàg paths endowed with the Borel sigma-field generated by Skorohod's topology, and $\xi = (\xi_t)_{t\geq 0}$ is the coordinate process, i.e. $\xi_t(\omega) = \omega(t)$. Our building block is a probability measure P on Ω for which ξ is a Lévy process, i.e. ξ has independent and homogeneous increments and starts from $\xi_0 = 0$ a.s. We write Π for the Lévy measure, which specifies the intensity of the jumps. We also denote by $\hat{P}$ the image of P by the map $\omega \to \hat{\omega} = -\omega$. In other words, $\hat{P}$ is the law of the dual Lévy process $\hat{\xi} = -\xi$ under P.

Killing the paths at their first-exit time from the upper half-line

$$T = \inf\{t \geq 0 : \xi_t \leq 0\}$$

yields two sub-Markovian transition probabilities on $(0,\infty)$

$$p_t(x, \mathrm{d}y) = P_x(\xi_t \in \mathrm{d}y, t < T) \text{ and } \hat{p}_t(x, \mathrm{d}y) = \hat{P}_x(\xi_t \in \mathrm{d}y, t < T)\,,$$

where P_x and $\hat{P}_x$ denote the law of $x + \xi$ under P and under $\hat{P}$, respectively. We write

$$U(x, \mathrm{d}y) = \int_0^\infty \mathrm{d}t p_t(x, \mathrm{d}y)$$

for the potential measure of the Lévy process killed when exiting $(0,\infty)$.

Recall that under P, the reflected process $(\sup_{0\leq s\leq t} \xi_s - \xi_t)_{t\geq 0}$ is a Feller process in $[0,\infty)$ which possesses a local time $(L_t)_{t\geq 0}$ at level 0. The (ascending) ladder time is defined as the right-continuous inverse of L, viz. $L^{-1}(t) = \inf\{s \geq 0 : L_s > t\}$ and the ladder height process H_+ by

$$H_+(t) = \xi_{L^{-1}(t)} = \sup_{0\leq s\leq L^{-1}(t)} \xi_s\,, \qquad \text{whenever } L^{-1}(t) < \infty\,.$$

Here, we use the convention $\inf \varnothing = \infty$ and $H_+(t) = \infty$ when $L_\infty \leq t$. It is well-known that H_+ is a subordinator (killed at time L_∞ when the Lévy process tends to $-\infty$). We denote its drift coefficient by $a_+ \geq 0$ and its Lévy measure by μ_+, so that for every $q, t \geq 0$,

$$E\left(\exp(-qH_+(t))\right) = \exp\left(-t\left(a_+q + \int_{(0,\infty]} \mu_+(\mathrm{d}x)(1 - \mathrm{e}^{-qx})\right)\right).$$

Here we agree that $\exp(-qH_+(t)) = 0$ when $L^{-1}(t) = \infty$, and $\mu_+(\{\infty\})$ corresponds to the killing rate of H_+. We write U_+ for its renewal function, viz.

$$U_+(x) = \int_0^\infty P(H_+(t) \leq x, L^{-1}(t) < \infty)\mathrm{d}t\,, \qquad x \in [0, \infty)\,.$$

We also consider the dual ladder H_-, that is the ladder height of the dual Lévy process $\hat{\xi}$ and denote by U_- its renewal function. According to Silverstein [18] (see also Theorem VI.20 in [2]), there is the remarkable identity

$$U(x, \mathrm{d}y) = \int_{(x-y)^+}^{x} U_-(\mathrm{d}z)U_+(\mathrm{d}y + z - x)\,. \tag{2}$$

More precisely, Silverstein's identity often appears with an additional constant factor $c > 0$ in the right-hand side of (2), which depends on the choice of the normalization that has been used to define the local times at 0 of the reflected Lévy processes. We thus implicitly assume that the local times have been normalized so that (2) holds exactly.

Silverstein [18] also observed that the renewal function U_- is harmonic for the semigroup induced by $(p_t)_{t\geq 0}$, i.e.

$$U_-(x) = \int_0^\infty p_t(x, \mathrm{d}y)U_-(y)\,, \qquad x > 0\,.$$

Following Doob, this enables us to construct (conservative) Markovian transition functions

$$p_t^\uparrow(x, \mathrm{d}y) = \frac{U_-(y)}{U_-(x)}p_t(x, \mathrm{d}y)\,.$$

The distribution of the Markov process on $(0, \infty)$ started from $x > 0$ and with transition functions $(p_t^\uparrow)_{t\geq 0}$ will be denoted by $P_x^\uparrow$; roughly speaking, $P_x^\uparrow$ should be viewed as the law of the Lévy process started from x and conditioned to stay positive. Indeed, if the Lévy process tends to ∞, that is $P_x(T = \infty) > 0$ for some (and then all) $x > 0$, then it is easily seen that there exists some constant $c' > 0$ such that $U_-(x) = c'P_x(T = \infty)$ and hence $p_t^\uparrow(x, \mathrm{d}y)$ coincides with the transition probability of the Lévy process conditioned to stay positive in the usual sense. Finally, we denote the potential measure of the Lévy process started from x and conditioned to stay positive by

$$U^{\uparrow}(x, \mathrm{d}y) = \int_0^{\infty} \mathrm{d}t p_t^{\uparrow}(x, \mathrm{d}y) = \frac{U_-(y)}{U_-(x)} U(x, \mathrm{d}y) \,. \tag{3}$$

When 0 is regular upwards, in the sense that $P(\sup_{0\le s\le\varepsilon} \xi_s > 0) = 1$ for any $\varepsilon > 0$, it is known from a work of Chaumont and Doney [7] that $P_x^{\uparrow}$ has a weak limit as $x \to 0+$, which we denote by $P_0^{\uparrow}$. More precisely $P_0^{\uparrow}(\xi_t > 0) = 1$ for all $t > 0$ and under $P_0^{\uparrow}$ the canonical process ξ remains Markovian (as a matter of fact, Fellerian) with transition probabilities $(p_t^{\uparrow})_{t\ge 0}$. We shall need the following result when the ladder height subordinator H_+ has a strictly positive drift coefficient (we refer to Vigon [19] for an explicit necessary and sufficient condition for this to happen).

Lemma 1 *Suppose that $a_+ > 0$. Then the following holds :*

(i) *The renewal function U_+ has a continuous derivative u_+ which is strictly positive everywhere with $u_+(0) = 1/a_+$. For every $x > 0$, the probability that the dual Lévy process started from x exits $(0, \infty)$ for the first time continuously is*

$$\hat{P}_x(\xi_T = \xi_{T-}, T < \infty) = a_+ u_+(x) \,.$$

Further, this quantity converges to $a_+/E(H_+(1))$ when $x \to \infty$.

(ii) *There is the identity*

$$U^{\uparrow}(0, \mathrm{d}y) = U_-(y) u_+(y) \mathrm{d}y \,.$$

Proof: The probability under $\hat{P}_x$ that ξ is continuous at its first exit time from $(0, \infty)$ coincides with the probability that the ladder height subordinator H_+ crosses the level x continuously. The first assertion in (i) merely rephrases a result of Neveu which is stated as Theorem III.5 in [2], while the last one is a consequence of the renewal theorem for subordinators (see, e.g. Proposition 3.3 in [3]).

So we focus on (ii). The existence of a regular density u_+ for the renewal function U_+ enables us to rewrite (2) in the form

$$U(x, \mathrm{d}y)/\mathrm{d}y = \int_{(x-y)^+}^{x} U_-(\mathrm{d}z) u_+(y + z - x) \,.$$

Combining with (3) readily yields the desired formula. □

2.2 A duality relation

In order to state the duality relation that lies at the heart of this work, we still need one more notation. We introduce the measure

$$m(\mathrm{d}x) = \overline{\Pi}(x)U_-(x)\mathrm{d}x + a_+\delta_0(\mathrm{d}x)\,, \qquad x \in [0,\infty) \tag{4}$$

where $\overline{\Pi}(x) = \Pi((x,\infty))$ is the (upper) tail distribution of the Lévy measure Π and a_+ the drift coefficient of the ladder height subordinator H_+.

Theorem 1 (i) *There is the duality identity*

$$p_t^{\uparrow}(x,\mathrm{d}y)U_-(x)\mathrm{d}x = \hat{p}_t(y,\mathrm{d}x)U_-(y)\mathrm{d}y\,.$$

(ii) *The duality measure $U_-(x)\mathrm{d}x$ can be expressed as*

$$U_-(x)\mathrm{d}x = \int_{[0,\infty)} m(\mathrm{d}y)U^{\uparrow}(y,\mathrm{d}x)\,.$$

Proof: (i) This follows immediately from Hunt's switching identity

$$p_t(x,\mathrm{d}y)\mathrm{d}x = \hat{p}_t(y,\mathrm{d}x)\mathrm{d}y$$

(see Theorem II.5 in [2]) and the definition of $p_t^{\uparrow}(x,\mathrm{d}y)$. Note that this has already been pointed out in the proof of Theorem 4 of Chaumont [6].

(ii) From (3) we get

$$\int_{[0,\infty)} \mathrm{d}y\overline{\Pi}(y)U_-(y)U^{\uparrow}(y,\mathrm{d}x) = U_-(x)\int_{[0,\infty)} \mathrm{d}y\overline{\Pi}(y)U(y,\mathrm{d}x).$$

On the other hand, Hunt's switching identity gives

$$\int_{[0,\infty)} \mathrm{d}y\overline{\Pi}(y)U(y,\mathrm{d}x) = \left(\int_{[0,\infty)} \hat{U}(x,\mathrm{d}y)\overline{\Pi}(y)\right)\mathrm{d}x$$

where

$$\hat{U}(x,\mathrm{d}y) = \int_0^{\infty} \mathrm{d}t\,\hat{p}_t(x,\mathrm{d}y)$$

is the potential measure of the dual Lévy process $\hat{\xi} = -\xi$ killed upon exiting $(0,\infty)$.

A standard argument based on the Poissonian structure of the jumps of the dual Lévy process and the compensation formula for Poisson point processes shows that

$$\int_{[0,\infty)} \hat{U}(x, \mathrm{d}y)\overline{\Pi}(y) = \hat{P}_x(\xi \text{ exits from } (0,\infty) \text{ by a jump}) .$$

See for instance the proof of Proposition III.2 in [2]. When the drift coefficient a_+ of the ladder height subordinator H_+ (under P) is zero, the probability above is one according to a celebrated result due to Kesten which is stated as Theorem II.4 in [2], and the proof is complete. When $a_+ > 0$, we deduce from Lemma 1 that

$$a_+ U^{\uparrow}(0, \mathrm{d}x)/\mathrm{d}x = U_-(x)\hat{P}_x(\xi \text{ exits from } (0,\infty) \text{ continuously }) ,$$

which yields the conclusion. □

2.3 Weak convergence of the over and under shoots

The probabilistic interpretation of the duality identity (Theorem 1) requires the measure m defined by (4) to be finite. The following claims are due to Vigon [19] (see (5.3.4) in [10]) and Doney and Maller (see Theorem 8 in [12]), respectively.

Lemma 2 *The mass of the measure m coincides with the mean ladder height, i.e.*

$$m([0,\infty)) = E(H_+(1)) .$$

This quantity is finite if and only if $\xi_1 \in L^1(P)$ and either $E(\xi_1) > 0$ or $E(\xi_1) = 0$ and

$$\int_{[1,\infty)} \mathrm{d}x \frac{x\overline{\Pi}(x)}{\int_0^x \mathrm{d}y \int_y^\infty \mathrm{d}z \Pi((-\infty, -z))} < \infty .$$

We assume that $E(H_+(1)) < \infty$ throughout the rest of this work, except at the beginning of Section 3.2. We stress that this rules out the case when the Lévy process tends to $-\infty$; in particular the ascending ladder processes are not defective.

Next, we introduce the probability measure ρ on $[0,\infty)^2$ as

$$\rho(\mathrm{d}x, \mathrm{d}y) = \frac{1}{E(H_+(1))}(U_-(x)\Pi(x + \mathrm{d}y)\mathrm{d}x + a_+\delta_0(\mathrm{d}x)\delta_0(\mathrm{d}y)) , \tag{5}$$

and write ρ_1 and ρ_2 for the marginal laws of ρ :

$$\begin{aligned}\rho_1(\mathrm{d}x) &= \frac{1}{E(H_+(1))}\left(U_-(x)\overline{\Pi}(x)\mathrm{d}x + a_+\delta_0(\mathrm{d}x)\right) ,\\ \rho_2(\mathrm{d}y) &= \frac{1}{E(H_+(1))}\left(a_+\delta_0(\mathrm{d}y) + \int_0^\infty \mathrm{d}xU_-(x)\Pi(x+\mathrm{d}y)\right) .\end{aligned}$$

Note that the first marginal ρ_1 coincides with the measure m in (4) normalized to be a probability, and that an integration by parts gives

$$\int_0^\infty \mathrm{d}xU_-(x)\Pi(x+\mathrm{d}y) = \left(\int_0^\infty U_-(\mathrm{d}x)\overline{\Pi}(x+y)\right)\mathrm{d}y\,.$$

According to Vigon's *équation amicale inversée* (see [19]), the right-hand side can be expressed as $\overline{\mu}_+(y)\mathrm{d}y$, where $\overline{\mu}_+$ denotes the tail of the Lévy measure of the ladder height subordinator H_+. Hence we also have

$$\rho_2(\mathrm{d}y) = \frac{1}{E(H_+(1))}\left(a_+\delta_0(\mathrm{d}y) + \overline{\mu}_+(y)\mathrm{d}y\right) ,$$

which is the classical limit distribution for the overshoot (i.e. the residual lifetime in the renewal process constructed from the subordinator H_+).

More precisely, it belongs to the folklore of fluctuation theory that when the ladder height of a random walk or a Lévy process has a finite mean, then the pair formed by the undershoot and the overshoot across a large level z converges weakly as $z \to \infty$; see in particular Theorem 3 in [15]. For every path $\omega \in \Omega$, let

$$\hat{T}(\omega) = T(\hat{\omega}) = \inf\{t \geq 0 : \omega(t) \in (0,\infty)\}$$

denote the first entrance time to the positive half-line. We now provide a formal statement of the convergence alluded above which stresses the role of the measure ρ defined in (5).

Lemma 3 *The probability measures on* $[0,\infty)^2$

$$\hat{P}_z(\xi_{T-} \in \mathrm{d}x, -\xi_T \in \mathrm{d}y) = P_{-z}(-\xi_{\hat{T}-} \in \mathrm{d}x, \xi_{\hat{T}} \in \mathrm{d}y)$$

converge to ρ *as* $z \to \infty$ *in the sense of weak convergence of probability measures.*

Proof: A by-product of the quintuple identity for first passage times for Lévy processes (see Doney and Kyprianou [11] and references therein for further results in this area) is that for

$x, y > 0$

$$P_{-z}(-\xi_{\hat{T}-} \in \mathrm{d}x, \xi_{\hat{T}} \in \mathrm{d}y) = \int_0^z U_+(z - \mathrm{d}v)\mathbf{1}_{\{x>v\}}U_-(\mathrm{d}x - v)\Pi(\mathrm{d}x + y)\,.$$

Roughly speaking, the renewal theorem implies that $U_+(z - \mathrm{d}v)$ converges as $z \to \infty$ towards $\mathrm{d}v/E(H_+(1))$, and this yields that

$$\lim_{z\to\infty} P_{-z}(-\xi_{\hat{T}-} \in \mathrm{d}x, \xi_{\hat{T}} \in \mathrm{d}y) = \frac{1}{E(H_+(1))}U_-(x)\Pi(x + \mathrm{d}y)\mathrm{d}x\,, \qquad \text{vaguely on } (0,\infty)^2.$$

This establishes the claim when the ladder height process has no drift, because the right-hand side then defines a probability measure on $(0,\infty)^2$. In the case when $a_+ > 0$, the same conclusion follows invoking further Lemma 1(i) and the Portemanteau theorem, as ρ is a probability measure. □

2.4 Time-reversal identities

On our way to providing the probabilistic interpretation of the duality relation of Theorem 1, we need to introduce some notation for càdlàg paths indexed by the entire real line.

We set $\overline{\Omega} = \mathbb{D}(\mathbb{R},\mathbb{R})$; $\overline{\omega}$ will denote a generic path in $\overline{\Omega}$. We also set $\xi_t(\overline{\omega}) = \overline{\omega}(t)$ for every $t \in \mathbb{R}$, so $(\xi_t)_{t\in\mathbb{R}}$ is the usual coordinate process. It may sometimes be convenient to identify $\overline{\Omega}$ as the product space $\Omega \times \Omega$ via the canonical bijection $\overline{\omega} \to (\omega, \omega')$, where

$$\omega(t) = -\omega(-t-) \text{ and } \omega'(t) = \omega(t) \qquad \text{for every } t \geq 0\,.$$

Equivalently, we have for $t \in \mathbb{R}$

$$\overline{\omega}(t) = \begin{cases} \omega'(t) \text{ if } t \geq 0\,, \\ -\omega(-t-) \text{ if } t < 0\,. \end{cases}$$

We then introduce a probability measure $\mathcal{P}$ on $\overline{\Omega}$ by

$$\mathcal{P}(\mathrm{d}\overline{\omega}) = \mathcal{P}(\mathrm{d}\omega, \mathrm{d}\omega') = \int_{[0,\infty)^2} \rho(\mathrm{d}x, \mathrm{d}y)P_x^{\uparrow}(\mathrm{d}\omega)P_y(\mathrm{d}\omega')\,.$$

Thus under $\mathcal{P}$, the pair $(-\xi_{0-}, \xi_0)$ has the stationary distribution ρ for the under and the over shoots, and conditionally on $(-\xi_{0-}, \xi_0) = (x, y)$, the processes $(-\xi_{t-})_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ are independent with laws $P_x^{\uparrow}$ and P_y, respectively. Note that $\xi_t < 0$ for $t < 0$ and $\lim_{t\to-\infty} \xi_t = -\infty$ $\mathcal{P}$-a.s. while for large times ξ oscillates or $\lim_{t\to\infty} \xi_t = \infty$ according as the genuine Lévy process oscillates or tends to ∞. In this section, we will essentially work with the canonical process on nonnegative times, $(\xi_t)_{t\geq 0}$, which has thus the law $P_{\rho_2} = \int \rho_2(\mathrm{d}x)P_x$ under $\mathcal{P}$.

We then fix a level $z > 0$ and let $\tau(z) = \inf\{t \in \mathbb{R} : \xi_t > z\}$ denote the first passage time above z. Note that $\tau(z) \in [0, \infty)$ $\mathcal{P}$-a.s. and that $\tau(z) = 0$ when $\xi_0 > z$. In the latter case, the notation $-\xi_{\tau(z)-}$ means $-\xi_{0-}$ (which is a nonnegative random variable), and $(\xi_t)_{0\leq t<\tau(z)}$ is the empty path.

Theorem 2 *The following assertions hold for each $z > 0$:*

(i) *We have the identity*

$$\mathcal{P}(z - \xi_{\tau(z)-} \in \mathrm{d}x, \xi_{\tau(z)} - z \in \mathrm{d}y) = \rho(\mathrm{d}x, \mathrm{d}y)\,.$$

(ii) *Under $\mathcal{P}$, the process $(\xi_t)_{0\leq t<\tau(z)}$ and the variable $\xi_{\tau(z)}$ are conditionally independent given $\xi_{\tau(z)-}$.*

(iii) *Under the conditional law $\mathcal{P}(\cdot \mid \xi_{\tau(z)-} = z - x)$, the process $(z - \xi_{(\tau(z)-t)-})_{0\leq t<\tau(z)}$ has the same law as the process $(\xi_t)_{0\leq t<\ell(z)}$ under $P^{\uparrow}_x$, where $\ell(z) = \sup\{t \geq 0 : \xi_t < z\}$ denotes the last exit-time from $(-\infty, z)$.*

Proof: (i) This just reflects the fact that ρ is the stationary distribution of the under and over shoots, as it can be seen from Lemma 3.

(ii) Consider a process $K = (K_t)_{t\geq 0}$ with nonnegative left-continuous paths, which is adapted to the natural filtration generated by ξ (hence K is predictable). Let also $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function with support in (z, ∞). We write $\Delta_t = \xi_t - \xi_{t-}$ for the jump of ξ at time t and $\mathcal{E}$ for the mathematical expectation under $\mathcal{P}$. By an application of the compensation formula to the Poisson point process of the jumps of a Lévy process at the second line below, we have

$$\begin{aligned}
\mathcal{E}\left(K_{\tau(z)} f(\xi_{\tau(z)})\right) &= \mathcal{E}\left(\sum_{t\geq 0} \mathbf{1}_{\{\sup_{0\leq s<t} \xi_s \leq z\}} K_t f(\Delta_t + \xi_{t-})\right) \\
&= \mathcal{E}\left(\int_0^\infty \mathrm{d}t \mathbf{1}_{\{\sup_{0\leq s<t} \xi_s \leq z\}} K_t \left(\int \Pi(\mathrm{d}x) f(x + \xi_{t-})\right)\right).
\end{aligned}$$

So if we define

$$\varphi(y) = \frac{1}{\overline{\Pi}(z-y)} \int \Pi(\mathrm{d}x) f(x+y) \qquad \text{for every } y \leq z$$

(we stress that $\varphi(z) = 0$ when $\Pi((0, \infty)) = \infty$, since our assumptions on f ensure that

$\int \Pi(\mathrm{d}x) f(x+z) < \infty$), we obtain

$$
\begin{aligned}
\mathcal{E}\left(K_{\tau(z)} f(\xi_{\tau(z)})\right) &= \mathcal{E}\left(\int_0^\infty \mathrm{d}t \mathbf{1}_{\{\sup_{0\le s<t} \xi_s \le z\}} K_t \varphi(\xi_{t-}) \overline{\Pi}(z-\xi_{t-})\right) \\
&= \mathcal{E}\left(\sum_{t\ge 0} \mathbf{1}_{\{\sup_{0\le s<t} \xi_s \le z\}} K_t \varphi(\xi_{t-}) \mathbf{1}_{\{\Delta_t + \xi_{t-} > z\}}\right) \\
&= \mathcal{E}\left(K_{\tau(z)} \varphi(\xi_{\tau(z)-})\right) ,
\end{aligned}
$$

where we again applied the compensation formula for Poisson point processes at the second line above. This establishes the claim of conditional independence.

(iii) To prove that under $P^{\uparrow}_{\rho_1} = \int \rho_1(\mathrm{d}x) P^{\uparrow}_x$ the process time-reversed at its last passage time below level z, $(\xi_{(\ell(z)-t)-} : 0 \le t < \ell(z))$, is sub-Markovian with semigroup $\hat{p}_t$ we shall apply Nagasawa's Theorem 3.5 in [17]. There are six conditions to be checked to ensure the validity of Nagasawa's result. To be rigorous and comply with the notation in [17] we put $E = (0,\infty)$ when $a_+ = 0$ and $E = [0,\infty)$ when $a_+ > 0$ (note that ρ_1 has mass at zero and ξ under $P^{\uparrow}_{\rho_1}$ can start from zero; however as mentioned before Lemma 1, $P^{\uparrow}_0(\xi_t > 0) = 1$ for each $t > 0$). We proceed by stating and demonstrating these conditions :

(1) Condition A 3.1 on p. 188 in [17] requires that the semigroups $p^{\uparrow}$ and $\hat{p}$ are in duality w.r.t. $\int_0^\infty U^{\uparrow}(x, dy)\rho_1(\mathrm{d}x)$. This holds due to Theorem 1 and $\rho_1(\mathrm{d}x) = m(\mathrm{d}x)/E(H_+(1))$.

(2) Condition A 3.2 and condition (ii) in the footnote on p. 190 in [17] are satisfied since our process is started from the probability measure ρ_1 and this ensures the finiteness of any quantities of the type $P^{\uparrow}_{\rho_1}(\xi_{(\ell(z)-t)-} \in .)$ and $\int_0^\infty e^{-\alpha t} P^{\uparrow}_{\rho_1}(\xi_{(\ell(z)-t)-} \in \cdot)\mathrm{d}t$, for $\alpha > 0$.

(3) Condition A 3.3 asks for right-continuity of $\hat{P}^T_t f(x) = \hat{E}_x(f(\xi_t), T > t)$ in t and a.s. right-continuity in t of $\int_0^\infty e^{-\alpha s} \hat{P}^T_s f(\xi_{(\ell(z)-t)-})\mathrm{d}s$ under any of the measures $P^{\uparrow}_a$ for $a \in E$, $x \in E$ and $\alpha > 0$, where f is a continuous function with a compact support in E. These are satisfied since the canonical process is defined in Ω.

(4) Condition (i) in the footnote on p. 190 in [17] can be read off as the existence of the left-limit $\xi_{\ell(z)-}$ on the event $\{0 < \ell(z) < \infty\}$, P_{ρ_1}-a.s. This plainly holds.

(5) Condition (iii) is similar to A 3.3. We need to check that $\hat{P}^T_t f(x)$ is continuous for $x \in E$ for any continuous function f with a compact support in E. However the statement is clear from the continuity of $\hat{P}_x(T > t)$ in x, for $x > 0$ and any fixed t, independently of $a_+ = 0$ or not, and the fact that when $a_+ > 0$, $\hat{T}$ is continuous at 0 since 0 is regular for $(0,\infty)$.

Thus, we have checked that $(\xi_{(\ell(z)-t)-} : 0 \le t < \ell(z))$ is sub-Markovian with semigroup $\hat{p}_t$. Note that the latter thus fulfills the strong Markov property and denote its initial distribution by

$$\nu_z(\mathrm{d}x) = P^{\uparrow}_{\rho_1}(\xi_{\ell(z)-} \in \mathrm{d}x, \ell(z) > 0) , \qquad \text{for } x \in [0, z].$$

Pick an arbitrary $z' > z$ and apply the strong Markov property to the time-reversed path $(\xi_{(\ell(z')-t)-} : 0 \le t < \ell(z'))$ at its first-passage time below level z, that is $\ell(z') - \ell(z)$. This yields

$$\nu_z(\mathrm{d}x) = \int_0^\infty \nu_{z'}(\mathrm{d}y)\hat{P}_{y-z}(z + \xi_T \in \mathrm{d}x)\,, \qquad \text{for } x \in [0, z].$$

Letting $z' \to \infty$, we deduce from Lemma 3 that $\nu_z(\mathrm{d}x) = \rho_2(z - \mathrm{d}x)$ on $[0, z]$.

We conclude that under $P^\uparrow_{\rho_1}$, the time-reversed process $(\xi_{(\ell(z)-t)-})_{0\le t<\ell(z)}$ has the same law as $(\xi_t)_{0\le t<T}$ under $\hat{P}_{\nu_z}$. Equivalently, under P_{ρ_2}, the process $(z - \xi_{(\tau(z)-t)-})_{0\le t<\tau(z)}$ has the same law as the process $(\xi_t)_{0\le t<\ell(z)}$ under $P^\uparrow_{\rho_1}$, which is our statement. □

In the special case when the drift coefficient a_+ of the ladder height subordinator H_+ is strictly positive, recall from Lemma 1 that the probability under $\hat{P}_x$ that the first-exit from $(0, \infty)$ occurs continuously equals $a_+ u_+(x) > 0$. Similarly, it is easy to see that for any $x > 0$, the probability under $P^\uparrow_0$ that the last-exit from $(0, x)$ occurs continuously is strictly positive. Hence we immediately deduce from Theorem 2 the following identity which can also be seen from Theorem 4 of Chaumont [6].

Corollary 1 *Suppose $a_+ > 0$ and fix $x > 0$. The law of $(\xi_t : 0 \le t < \ell(x))$ under $P^\uparrow_0(\cdot \mid \xi$ is continuous at $\ell(x))$ is that of $(\xi_{(T-t)-} : 0 \le t < T)$ under $\hat{P}_x(\cdot \mid \xi$ is continuous at $T)$.*

Remark. Note that these conditionings are trivial when ξ has no positive jumps under P, and hence Corollary 1 encompasses the extension of Williams' time-reversal stated as Theorem VII.18 in [2].

In the same vein, we recover (and slightly extend) a result due to Duquesne; see Theorems 4.1 and 4.2 in [13].

Corollary 2 *Under $P^\uparrow_0$ the law of $(\xi_{\ell(x)-} - \xi_{(\ell(x)-t)-})_{t<\ell(x)}$ is the law of $(\xi_t)_{t<g(\hat{T}(x))}$ under P, where $\hat{T}(x) = \inf\{t \ge 0 : \xi_t > x\}$ and $g(\hat{T}(x)) = \sup\{s < \hat{T}(x) : \xi_s \vee \xi_{s-} = \sup_{t<\hat{T}(x)} \xi_t\}$.*

Remark. Note that Corollary 2 is intuitively obvious when we deal with random walks because the law $P^\uparrow_0$ equals the law under P of the reversed excursions away from the maximum.

Proof: Recall that we assume that $E(H_+(1)) < \infty$ and denote by $I = \inf_{t\ge 0} \xi_t$ and $g_I = \sup_{s\ge 0}\{\xi_s \wedge \xi_{s-} = I\}$. We deduce from Theorem 25 in Chapter 8 in [10] that under $P^\uparrow_{\rho_1}$ the process $(\xi_{g_I+t} - \xi_{g_I})_{t\ge 0}$ has the law of $(\xi_t)_{t\ge 0}$ under $P^\uparrow_0$ and it is independent of $(\xi_t)_{t\le g_I}$. This combined with Theorem 2 yields that for every bounded measurable functional F on Ω

$$\begin{aligned} E^\uparrow_{\rho_1}\left[F\big((\xi_{\ell(x)-} - \xi_{(\ell(x)-t)-})_{t<\ell(x)-g_I}\big)\right] &= E^\uparrow_0\left[F\big((\xi_{\ell(x-I)-} - \xi_{(\ell(x-I)-t)-})_{t<\ell(x-I)}\big)\right] \\ &= E\left[F\big((\xi_t)_{t<g(T(x-\rho_2))}\big)\right], \end{aligned} \qquad (6)$$

where $g(T(x-\rho_2))$ is the time when the last maximum is attained before ξ goes beyond $x-\rho_2$, where ρ_2 is the stationary overshoot and is independent of ξ under P and I in the second term has the distribution of $P^{\uparrow}_{\rho_1}(I \in dy)$ and is independent of ξ under $P^{\uparrow}_0$. Next note that Theorem 2 implies as well that

$$P^{\uparrow}_{\rho_1}(I \in dy) = \lim_{z\to\infty} \hat{P}_{\rho_1}(\inf_{s<\ell(z)} \xi_s \in dy) = \lim_{z\to\infty} P_{\rho_2}(\inf_{s<\tau(z)} z - \xi_{\tau(z-s)-} \in dy) = \rho_2(dy).$$

The last identity can be deduced as in Lemma 3 or recovered from Theorem 3 in [15]. Then (6) translates easily to $A * \rho_2(x) = B * \rho_2(x)$ with $B(y) = EF\big((\xi_t)_{t<g(T(y))}\big)$ and $A(y) = E^{\uparrow}_0 F\big((\xi_{\ell(y)-} - \xi_{(\ell(y)-t)-})_{t<\ell(y)}\big)$ and we get that $A = B$ on $(0,\infty)$. Since this holds for any measurable functional F we conclude the proof. □

3 Applications to weak limit theorems

3.1 Starting a Lévy process from $-\infty$

We first observe that the probability measure $\mathcal{P}$ that has been introduced in Section 2.4 fulfills a remarkable spatial stationarity property, which follows easily from Theorem 2 and the strong Markov property.

Corollary 3 *For any $x \in \mathbb{R}$, let $\tau(x) = \inf\{t \in \mathbb{R} : \xi_t > x\}$ denote the first passage time of ξ above the level x. Under $\mathcal{P}$, the processes $(\xi_{\tau(x)+t})_{t\in\mathbb{R}}$ and $(x+\xi_t)_{t\in\mathbb{R}}$ have the same distribution.*

Proof: Let $\mathcal{P}_z$ be the law of $(z+\xi_t)_{t\in\mathbb{R}}$ for an arbitrary $z \in \mathbb{R}$. An application of the strong Markov property combined with Theorem 2 shows the following. Let us work under $\mathcal{P}_z$ for an arbitrary $z < 0$, and recall that $\hat{T}$ denotes the first entrance time into $(0,\infty)$. Then the pair $(-\xi_{\hat{T}-}, \xi_{\hat{T}})$ has the law ρ, and conditionally on $(-\xi_{\hat{T}-}, \xi_{\hat{T}}) = (x,y)$, the processes $(-\xi_{(\hat{T}-t)-})_{0\le t<\hat{T}}$ and $(\xi_{t+\hat{T}})_{t\ge 0}$ are independent. Further the former has the same law as $(\xi_t)_{0\le t<\ell(-z)}$ under $P^{\uparrow}_x$ while the latter has the law P_y. As the last-exit time from $(-\infty,-z)$, $\ell(-z)$, tends to infinity as $z \to -\infty$, we see that the law of the shifted process $(\xi_{t+\hat{T}})_{t\in\mathbb{R}}$ under $\mathcal{P}_z$ converges weakly to that of $(\xi_t)_{t\in\mathbb{R}}$ under $\mathcal{P}$. We can now complete the proof by an easy argument based on replacing z by $z+x$. □

The proof of Corollary 3 suggests that the limit theorem for the under and over shoots should have an extension to paths. In this direction, for every $\omega \in \mathbb{D}([0,\infty),\mathbb{R})$, we denote by $\vartheta(\omega)$ the path indexed by the entire real line which is obtained by shifting ω at the time $\hat{T}(\omega) = \hat{T}$

of its first entrance in $(0,\infty)$, that is $\vartheta(\omega) = (\omega'(s))_{s\in\mathbb{R}}$ with

$$\omega'(s) = \begin{cases} \omega(\hat{T}+s) \text{ for every } s \geq -\hat{T}\,, \\ \qquad -\infty \text{ otherwise}\,. \end{cases}$$

We now state the main result of this section.

Theorem 3 *Fix any $b \in \mathbb{R}$. The law of $(\xi_t \circ \vartheta)_{t\geq b}$ under P_x converges weakly on $\mathbb{D}([b,\infty),\mathbb{R})$ as $x \to -\infty$ towards the law of $(\xi_t)_{t\geq b}$ under $\mathcal{P}$.*

We shall derive Theorem 3 from Corollary 3 using a coupling argument which requires distinguishing whether the drift coefficient a_+ of the ladder height subordinator H_+ is zero or strictly positive. The first case is easier and relies on the following standard construction.

Lemma 4 *Fix $\varepsilon > 0$. We can construct on some probability space a random variable γ with values in $[0,\varepsilon]$, an a.s. finite random time τ and a pair of processes $(\xi'_t)_{t\geq 0}$ and $(\xi''_t)_{t\geq 0}$ that fulfill the following requirements :*

(i) *ξ' has the law P and ξ'' has the law P_{ρ_2},*

(ii) *$\xi''_t = \xi'_t + \gamma$ for all $t \geq \tau$.*

Proof: We start from a pair $(\tilde{\xi}', \xi'')$ with law $P \otimes P_{\rho_2}$, i.e. $\tilde{\xi}'$ and ξ'' are independent with respective laws P and P_{ρ_2}. Then $\xi'' - \tilde{\xi}'$ is a symmetric Lévy process with initial law ρ_2. Recall from Lemma 2 that it is centered and hence recurrent by the test of Chung and Fuchs (cf. Exercise I.10 in [2]). We set $\tau = \inf\{t \geq 0 : \xi''_t - \tilde{\xi}'_t \in [0,\varepsilon]\}$ and $\gamma = \xi''_\tau - \tilde{\xi}'_\tau$, so τ is an a.s finite stopping time and γ a random variable in $[0,\varepsilon]$. By the strong Markov property, the process

$$\xi'_t = \begin{cases} \quad \tilde{\xi}'_t \ \text{ if } t \leq \tau \\ \xi''_t - \gamma \ \text{ if } t > \tau \end{cases}$$

has the law P. $\square$

When the drift coefficient of the ladder height subordinator H_+ is strictly positive, $a_+ > 0$, we need a stronger coupling.

Lemma 5 *Assume $a_+ > 0$. We can construct on some probability space a pair of a.s. finite random times τ' and τ'' and a pair of processes $(\xi'_t)_{t\geq 0}$ and $(\xi''_t)_{t\geq 0}$ that fulfill the following requirements :*

(i) *ξ' has the law P and ξ'' has the law P_{ρ_2},*

(ii) *$\xi'_{\tau'+t} = \xi''_{\tau''+t}$ for all $t \geq 0$.*

Proof: We start again from a pair $(\tilde{\xi}', \xi'')$ with law $P \otimes P_{\rho_2}$. We define the passage times

$$\sigma_1' = \inf\{t \geq 0 : \tilde{\xi}_t' \geq \xi_0''\} \ , \ \sigma_1'' = \inf\{t \geq 0 : \xi_t'' \geq \tilde{\xi}'(\sigma_1')\}$$

and then recursively

$$\sigma_{k+1}' = \inf\{t \geq 0 : \tilde{\xi}_t' \geq \xi''(\sigma_k'')\} \ , \ \sigma_{k+1}'' = \inf\{t \geq 0 : \xi_t'' \geq \tilde{\xi}'(\sigma_{k+1}')\} \, .$$

We claim that a.s., these non-decreasing sequences remain constant after a finite number of steps. Taking this assertion for granted, the construction of the coupling is immediate as it suffices to set $\tau' = \sigma_\infty'$, $\tau'' = \sigma_\infty''$, and

$$\xi_t' = \begin{cases} \tilde{\xi}_t' & \text{if } t \leq \tau' \\ \xi_{\tau''+t-\tau'}'' & \text{if } t > \tau' \, . \end{cases}$$

To complete the proof, it suffices to recall from Lemma 1 (i) that, since the ladder height subordinator H_+ has a strictly positive drift coefficient and a finite mean, the probability that H_+ hits some fixed point $x \in [0, \infty)$ is bounded from below by a strictly positive constant. Hence the number of steps alluded above is stochastically bounded by a geometric variable. $\square$

We may now tackle the proof of Theorem 3.

Proof: Recall the notation $\mathcal{P}_x$ for the law of $x + \xi$ under $\mathcal{P}$

1. Suppose first $a_+ > 0$ and fix $\varepsilon > 0$ arbitrarily small. According to Lemma 5, we can construct two process $(\xi_t')_{t \geq 0}$ and $(\xi_s'')_{s \in \mathbb{R}}$ with respective laws P and $\mathcal{P}$ and two a.s. finite random times τ' and τ'' such that $\xi_{\tau'+t}' = \xi_{\tau''+t}''$ for all $t \geq 0$. Provided that x is chosen sufficiently large, the probability of the event that

$$\sup_{0 \leq t \leq \tau'} \xi_t' \leq x/2 \ , \ \sup_{t \leq \tau''} \xi_t'' \leq x/2 \text{ and } \inf\{t \geq 0 : \xi_{t+\tau'}' - \xi_{\tau'}' > x/2\} > -b$$

is at least $1 - \varepsilon$.

Therefore if we set $\tilde{\xi}_t' = \xi_t' - x$ and $\tilde{\xi}_t'' = \xi_t'' - x$, then the processes $(\tilde{\xi}_t')_{t \geq 0}$ and $(\tilde{\xi}_s'')_{s \in \mathbb{R}}$ have the law P_{-x} and $\mathcal{P}_{-x}$, respectively. Further the probability that the paths obtained by shifting $\tilde{\xi}'$ and $\tilde{\xi}''$ at their first entrance time into $(0, \infty)$ coincide on $[b, \infty)$ is bounded from below by $1 - \varepsilon$. This entails the statement as we know from Corollary 3 that the path obtained by shifting $\tilde{\xi}''$ has the law $\mathcal{P}$.

2. Suppose now that $a_+ = 0$ and fix $\eta > 0$ arbitrarily small. Then the stationary distribution ρ_2 of the overshoot has no atom at 0, and we may pick $\varepsilon > 0$ sufficiently small so that $\rho_2([0, \varepsilon]) < \eta/2$. According to Lemma 4, we can construct two process $(\xi_t')_{t \geq 0}$ and $(\xi_s'')_{s \in \mathbb{R}}$ with respective

laws P and $\mathcal{P}$, an a.s. finite random time τ and a random variable $\gamma \in [0,\varepsilon]$ such that $\xi'_t + \gamma = \xi''_t$ for all $t \geq \tau$.

For every $x \geq 0$, consider the first entrance times

$$\tau'(x) = \inf\{t \geq 0 : \xi'_t > x\} \text{ and } \tau''(x) = \inf\{t \geq 0 : \xi''_t > x\}\,.$$

Since $\xi'_t \leq \xi''_t \leq \xi'_t + \varepsilon$ for all $t > \tau$, the probability of the event

$$\{\tau < \tau'(x) \wedge \tau''(x), \tau'(x) \neq \tau''(x)\}$$

is bounded from above by the probability that the overshoot $\xi''_{\tau''(x)} - x$ does not exceed ε, and thus by $\eta/2$. Further, provided that x is chosen sufficiently large, the probability of the event $\{\tau - b < \tau'(x) \wedge \tau''(x)\}$ is at least $1 - \eta/2$.

Now set $\tilde{\xi}'_t = \xi'_t - x$ and $\tilde{\xi}''_t = \xi''_t - x$; the processes $(\tilde{\xi}'_t)_{t\geq 0}$ and $(\tilde{\xi}''_s)_{s\in\mathbb{R}}$ have thus the law P_{-x} and $\mathcal{P}_{-x}$, respectively. It follows from above that the probability that the paths obtained by shifting $\tilde{\xi}'$ and $\tilde{\xi}''$ at their first entrance time into $(0,\infty)$ remain parallel on the time interval $[b,\infty)$ with a distance at most ε, is bounded from below by $1-\eta$. This entails the statement as we know from Corollary 3 that the path obtained by shifting $\tilde{\xi}''$ has the law $\mathcal{P}$. □

3.2 A Lamperti-type representation for self-similar Markov processes entering from $0+$

We now conclude this work with an application to the class of Markov processes in $(0,\infty)$ that enjoy the scaling property. That is, we consider a Markov process $X = (X_t)_{t\geq 0}$ with values in $(0,\infty)$ and write $\mathbb{P}_x$ for its law started from $X_0 = x > 0$. We shall always assume that the process is conservative, i.e. there is no cemetery state. We suppose that the self-similarity property

$$\text{the distribution of } (cX_{t/c})_{t\geq 0} \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}$$

holds for every $c, x > 0$. Lamperti [16] has studied in depth this class of processes which are nowadays called positive self-similar Markov processes (in short pssMp), and pointed at a fundamental connection with real valued Lévy processes that can be described as follows in the framework of this paper.

We work under the probability measure P_y for which $(\xi_t)_{t\geq 0}$ is a Lévy process started from $y \in \mathbb{R}$. We drop for a moment the assumption that the ladder height has a finite expectation, and just suppose that the Lévy process does not tend to $-\infty$. We introduce a time-change $\gamma(t)$

for every $t \geq 0$ as the inverse of the exponential functional, that is

$$\int_0^{\gamma(t)} \mathrm{e}^{\xi_s} \mathrm{d}s = t\,.$$

Then the process $X_t = \exp(\xi_{\gamma(t)})$ is a pssMp started from $x = \mathrm{e}^y$, and any (conservative) pssMp can be constructed by this procedure.

The question of whether a pssMp can enter from the boundary point $0+$, that is if $\mathbb{P}_x$ admits a non-degenerate weak limit $\mathbb{P}_{0+}$ as $x \to 0+$ was raised by Lamperti. Bertoin and Yor [4] provided a positive answer when the underlying Lévy process possesses a positive and finite first moment. Recall that this implies that the mean ladder height $E(H_+(1))$ is finite; it is further easy to show that a pssMp cannot enter from $0+$ when $E(H_+(1)) = \infty$. Caballero and Chaumont [5] obtained an explicit necessary and sufficient condition; basically they proved that a pssMp can enter from $0+$ if and only if $E(H_+(1)) < \infty$ and some very mild technical condition holds. That this technical condition is automatically fulfilled when the mean ladder height is finite has been proved recently by Chaumont *et al.* [8], so the definitive simple characterization is that a pssMp can enter from $0+$ if and only if $E(H_+(1)) < \infty$. ¿From now on this assumption is thus again enforced.

We point at a simple and direct construction of the law $\mathbb{P}_{0+}$ in terms of the spatially homogeneous law $\mathcal{P}$ and the canonical process $(\xi_t)_{t\in\mathbb{R}}$ indexed by the whole real line, which is somehow hidden in the approach by Caballero and Chaumont [5]. The intuition stems from the observation that Lamperti's transformation can be re-expressed in terms of the shifted path $\xi \circ \vartheta$: if we write $\sigma : [0, \infty) \to \mathbb{R}$ for the inverse of the functional

$$t \to \int_{-\infty}^{t} \exp(\xi_s \circ \vartheta)\mathrm{d}s\,, \qquad t \in \mathbb{R}\,,$$

then for every $y \in \mathbb{R}$, under P_y the time-changed process $(\exp(\xi_{\sigma(t)} \circ \vartheta))_{t\geq 0}$ has the law $\mathbb{P}_x$ with $x = \mathrm{e}^y$. Since we know from Theorem 3 that the law of the shifted path $\xi \circ \vartheta$ under P_y converges weakly to $\mathcal{P}$, we arrive naturally at the following.

Corollary 4 (i) *The exponential functional*

$$I(t) = \int_{-\infty}^{t} \mathrm{e}^{\xi_s} \mathrm{d}s$$

is finite for all $t \in \mathbb{R}$ and $I(\infty) = \infty$, $\mathcal{P}$-a.s.

(ii) *Introduce the time-change $\sigma(t)$ for $t > 0$ by*

$$\int_{-\infty}^{\sigma(t)} \mathrm{e}^{\xi_s} \mathrm{d}s \ = t\,.$$

Then under $\mathcal{P}$, the process $(X_t)_{t>0}$ given by

$$X_t = \exp(\xi_{\sigma(t)})$$

is a pssMp started from the entrance boundary $0+$, in the sense that its law is the weak limit of $\mathbb{P}_x$ as $x \to 0+$.

Proof: (i) By Theorem 1(ii), we have

$$\mathcal{E}(I(0)) = \int_0^\infty \rho_1(\mathrm{d}x) \int_0^\infty U^\uparrow(x, \mathrm{d}y)\mathrm{e}^{-y} = \frac{1}{E(H_+(1))} \int_0^\infty \mathrm{d}y \mathrm{e}^{-y} U_-(y)\,.$$

Because a renewal function is always sub-additive, the right-hand side is finite, which implies our claim.

(ii) We now know that $X_t = \exp(\xi_{\sigma(t)})$ is well-defined. Recall that $\tau(y) = \inf\{s \in \mathbb{R} : \xi_s > y\}$ denotes the first entrance time in (y, ∞), so $I(\tau(y))$ corresponds to the first passage time of X above the level $x = \mathrm{e}^y$. It follows readily from Corollary 3 and Lamperti's transformation that conditionally on $X_{I(\tau(y))} = z$, the shifted process $(X_{t+I(\tau(y))})_{t\geq 0}$ has the law $\mathbb{P}_z$. Further the distribution of $X_{I(\tau(y))}$, say μ_x, is the image of ρ_2 by the map $w \to x\mathrm{e}^w$. Plainly μ_x converges to δ_0 as $x \to 0+$ (i.e. $y \to -\infty$), while $\tau(y) = \sigma(I(\tau(y)) \to -\infty$. We conclude that $(X_t)_{t>0}$ has the law $\mathbb{P}_{0+}$. □

We stress that the argument for establishing Corollary 4 has little to do with self-similarity or exponential functions. It would apply just as well to construct the Markov process entering from the boundary $0+$ in the following more general situation (which mirrors Feller's construction of one-dimensional diffusions as time-space transforms of Brownian motions). More precisely, consider a measurable locally bounded function $f : \mathbb{R} \to (0, \infty)$ with $\int_{-\infty}^0 \mathrm{d}y|y|f(y) < \infty$ and $\int_0^\infty \mathrm{d}y f(y) = \infty$, and $g : \mathbb{R} \to (0, \infty)$ a continuous strictly increasing function with $\lim_{z\to-\infty} g(z) = 0$. Then the functional

$$I(t) = \int_{-\infty}^t f(\xi_s)\mathrm{d}s$$

is finite for all $t \in \mathbb{R}$ and $\lim_{t\to\infty} I(t) = \infty$, $\mathcal{P}$-a.s. Writing σ for the inverse functional of I, the process $(X_t)_{t>0}$ defined by $X_t = g(\xi_{\sigma(t)})$ then enters from $0+$, and is Markovian in $(0, \infty)$ with an infinitesimal generator defined by an obvious transformation of that of the Lévy process.

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References

[1] Azéma, J. Théorie générale des processus et retournement du temps. *Ann. Sci. École Norm. Sup.* (4) **6** (1973), 459–519 (1974).

[2] Bertoin, J. *Lévy processes.* Cambridge Tracts in Mathematics, 121. Cambridge University Press, Cambridge, 1996.

[3] Bertoin, J. *Subordinators: examples and applications.* Lectures on probability theory and statistics (Saint-Flour, 1997), 1–91, Lecture Notes in Math. **1717**, Springer, Berlin, 1999.

[4] Bertoin, J., and Yor, M. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.* **17** (2002), no. 4, 389–400.

[5] Caballero, M.E., and Chaumont, L. Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.* **34** (2006), no. 3, 1012–1034.

[6] Chaumont, L. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.* **64** (1996), no. 1, 39–54.

[7] Chaumont, L., and Doney, R.A. On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10** (2005), no. 28, 948–961.

[8] Chaumont, L. , Kyprianou, A. , Pardo, J.C. and Rivero, V. Fluctuation theory and exit systems for positive self-similar Markov processes. Preprint, 2009.

[9] Chung, K.L. and Walsh, J.B. To reverse a Markov process. *Acta Math.* **123** (1969) 225–251.

[10] Doney, R.A. *Fluctuation theory for Lévy processes.* Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. Lecture Notes in Mathematics, **1897**. Springer, Berlin, 2007.

[11] Doney, R.A., and Kyprianou, A. E. Overshoots and undershoots of Lévy processes. *Ann. Appl. Probab.* **16** (2006), no. 1, 91–106.

[12] Doney, R.A., and Maller, R.A. Stability of the overshoot for Lévy processes. *Ann. Probab.* **30** (2002), no. 1, 188–212.

[13] Duquesne, T. Path decompositions for real Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), no. 2, 339–370.

[14] Kyprianou, A.E. *Introductory lectures on fluctuations of Lévy processes with applications.* Universitext. Springer-Verlag, Berlin, 2006.

[15] Kyprianou, A.E., Pardo, J.C. and Rivero, V. Exact and asymptotic n-tuple laws at first and last passage time. To appear in *Ann. Appl. Probab.* .

[16] Lamperti, J. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **22** (1972), 205–225.

[17] Nagasawa, M. Time reversions of Markov processes. *Nagoya Math. J.* **24** (1964), 177–204.

[18] Silverstein, M.L. Classification of coharmonic and coinvariant functions for a Lévy process. *Ann. Probab.* **8** (1980), no. 3, 539–575.

[19] Vigon, V. Votre Lévy rampe-t-il? *J. London Math. Soc.* (2) **65** (2002), no. 1, 243–256.

[20] Williams, D. Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc.* (3) **28** (1974), 738–768.